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Gleason's Theorem and Cauchy's Functional Equation

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We study measures on the effect algebras of the closed interval [0, 1] and we describe regular or bounded measures. Applying Gleason's theorem for measures on the system of all closed subspaces of a Hilbert space, we show that any bounded measure m has the form $m(t) = tm(1)$, $t \in [0, 1]$, and as a by-product it gives a solution of Cauchy's basic functional equation $f(x + y) = f(x) + f(y)$ for $x, y, x + y \in [0, 1]$.

1. EFFECT ALGEBRAS AND MEASURES ON [0, 1]

The study of the mathematical foundations of quantum mechanics involves many interesting mathematical structures such as quantum logics, orthomodular posets, orthomodular lattices, and orthoalgebras. An important example of these structures is the system $L(H)$ of all closed subspaces of a real or complex Hilbert space H. To describe unsharp measurements in quantum mechanics (Busch *et al.*, 1991), we use the system $\mathscr{E}(H)$ of all effects, i.e., of all Hermitian operators A on a Hilbert space H such that $O \leq A \leq I$, where I is the identity on H.

Recently there appeared a new axiomatic model (Giuntini and Greuling, 1989; Kôpka and Chovanec, 1994; Foulis and Bennett, 1994), *effect algebras*, which describe both algebraic and unsharp properties of a propositional system.

An *effect algebra* is a nonempty set L with two particular elements 0, 1 and with a partial binary operation $\oplus$: $L \times L \to L$ such that for all a, b, $c \in L$ we have:

(EAi) If $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity).

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(EAii) If $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity).

(EAiii) For any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation).

(EAiv) If $1 \oplus a$ is defined, then $a = 0$ (zero–one law).

Let a and b be two elements of an effect algebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less than or equal to* b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$, and $b = a^{\perp}$.

If $a \leq b$, for the element c in (ii) with $a \oplus c = b$ we write $c = b \ominus a$, and c is called the *difference* of a and b. It is evident that

$$b \ominus a = (a \oplus b^{\perp})^{\perp}$$

An *atom* of L is a nonzero element $a \in L$ such that if $b \leq a$ for $b \in L$, then either $b = a$ or $b = 0$. An effect algebra L is said to be atomic if for any $a \in L \setminus \{0\}$ there exists an atom b of L such that $b \leq a$.

Let L be an effect algebra. Let $F = \{a_1, \ldots, a_n\}$ be a finite sequence in L. Recursively we define for $n \geq 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n \tag{1.1}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in effect algebras we conclude that (1.1) is correctly defined. By definition we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$ and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and any k with $1 \leq k \leq n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n} \tag{1.2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n) \tag{1.3}$$

We say that a finite sequence $F = \{a_1, \ldots, a_n\}$ in L is $\oplus$-*orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has a $\oplus$-*sum*, $\bigoplus_{i=1}^{n} a_i$ defined via

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n \tag{1.4}$$

It is clear that two elements a and b of L are orthogonal, i.e., $a \perp b$, iff $\{a, b\}$ is $\oplus$-orthogonal.

An arbitrary system $G = \{a_i\}_{i \in L}$ of not necessarily different elements of L is $\oplus$-*orthogonal* iff, for every finite subset F of I, the system $\{a_i\}_{i \in F}$ is

$\oplus$-orthogonal. If $G = \{a_i\}_{i \in I}$ is $\oplus$-orthogonal, so is any $\{a_i\}_{i \in J}$ for any $J \subseteq I$. An $\oplus$-orthogonal system $G = \{a_i\}_{i \in I}$ of L has a $\oplus$-*sum* in L, written as $\bigoplus_{i \in I} a_i$, iff in L there exists the join

$$\bigoplus_{i \in I} a_i := \bigvee_F \bigoplus_{i \in F} a_i \tag{1.5}$$

where F runs over all finite subsets in I. In this case, we also write $\oplus G := \bigoplus_{i \in I} a_i$.

It is evident that if $G = \{a_1, \ldots, a_n\}$ is $\oplus$-orthogonal, then the $\oplus$-sums defined by (1.4) and (1.5) coincide.

It is well known that any Boolean algebra, orthomodular lattice, and orthomodular poset can be organized into an effect algebra supposing that $a \oplus b$ is defined iff $a \perp b$ and then $a \oplus b := a \vee b$. Similarly, any orthoalgebra is an effect algebra.

Two prototypes of effect algebras are the following two examples.

Example 1.1. The set $\mathscr{E}(H)$ of all Hermitian operators A on H such that $O \leq A \leq I$, where I is the identity operator on H, is an effect algebra (which is not an orthoalgebra); a partial ordering $\leq$ is defined via $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$, and $C = A \oplus B$ iff $(Ax, x) + (Bx, x) = (Cx, x)$, $x \in H$. Any $\oplus$-orthogonal system has the sum in $\mathscr{E}(H)$, and $\mathscr{E}(H)$ is not a lattice.

Example 1.2. Let the closed interval $[0, 1]$ be ordered in the natural way, and for two numbers $a, b \in [0, 1]$, we define $a \oplus b$ iff $a + b \leq 1$ and we put then $a \oplus b = a + b$. Then $[0, 1]$ is a totally ordered distributive lattice in that any $\oplus$-orthogonal system has the sum in it. We recall that $\{a_s\}$ is $\oplus$-orthogonal iff $\{a_s\}$ is summable and $\Sigma_s a_s \leq 1$; then $\oplus_s a_s = \Sigma_s a_s$.

A real-valued mapping m on an orthoalgebra L is said to be a (finitely additive) *measure* if

$$m(a \oplus b) = m(a) + m(b), \qquad a, b \in L$$

It is clear that $m(0) = 0$.

If for a mapping m: $L \to \mathrm{R}$ we have

$$m\left(\bigoplus_{i \in I} a_i\right) = \sum_{i \in I} m(a_i) \tag{1.6}$$

whenever $\bigoplus_{i \in I} a_i$ exists in L, m is said to be a σ-*additive* or *completely additive measure* if (1.6) holds for any countable or any index set I, respectively. If a measure m is positive, then

$$m(a) \leq m(b) \qquad \text{whenever} \quad a \leq b$$

A positive measure m with $m(1) = 1$ is said to be a *state*; the system of all states on L is denoted by $\Omega(L)$. A measure m is said to be (i) *Jordan* if there are two positive measures m_1 and m_2 on L such that $m = m_1 - m_2$; (ii) *bounded* if $\sup\{|m(a)|: a \in L\} < \infty$.

2. MEASURES ON [0, 1]

In this section, we concentrate on measures on the effect algebra $L = [0, 1]$. Let $\mathcal{H}$ be a nonvoid subset of [0, 1]. We say that a measure m on [0, 1] is *$\mathcal{H}$-regular* if given $a \in [0, 1]$ and given $\epsilon > 0$ there exists $b \in \mathcal{H}$ with $b \leq a$ such that $|m(a) - m(b)| < \epsilon$.

(1) Let Q_1 be the set of all rational numbers in [0, 1]. It is simple to show that if m is a measure on [0, 1] and if $q \in Q_1$, then

$$m(q) = qm(1), \qquad q \in Q_1 \tag{2.1}$$

Indeed, $m(1) = m(\oplus_{i=1}^{n} 1/n) = nm(1/n)$, so that $m(1/n) = (1/n)m(1)$. Similarly we have $m(l/n) = m(\oplus_{i=1}^{l} 1/n) = lm(1/n) = (l/n)m(1)$. In an analogous way,

$$m(qt) = qm(t) \tag{2.2}$$

for all $t \in [0, 1]$, $q \in Q_1$ with $qt \in [0, 1]$.

(2) Hence, if m is positive (or negative), then m is monotone, and given $t \in (0, 1)$, there are sequences of rational numbers $\{q_n\}$ and $\{r_n\}$ from Q_1 such that $q_n \nearrow t$, $r_n \searrow t$. Then

$$q_n m(1) = m(q_n) \leq m(t) \leq m(r_n) = r_n m(1)$$

so that

$$m(t) = tm(1), \qquad t \in [0, 1] \tag{2.3}$$

(3) Similarly, if m is nondecreasing (nonincreasing), then m is of the form (2.3), and m is positive (negative). Therefore, any Jordan measure m on [0, 1] is of the form (2.3).

(4) Let m be nonnegative (nonpositive) for sufficiently small positive values of s, i.e., there is an interval $[0, \delta]$ such that $f(s) \geq 0$ [$f(s) \leq 0$] on $[0, \delta]$. Then, for any $t \in (0, 1)$ and any $s \in [0, \delta]$ with $t + s \leq 1$, we have

$$m(t \oplus s) = m(t) + m(s) \geq m(t), \qquad t \in (0, 1)$$

which means that m is nondecreasing. Hence, m is of the form (2.3).

We now formulate the main result of the paper. It is interesting that for the implication (x) $\Rightarrow$ (ix) we use the Gleason theorem.

Theorem 2.1. Let m be a measure on [0, 1]. The following statements are equivalent:

(i) $m(t) = tm(1)$, $t \in [0, 1]$.

(ii) m is Q_1-regular.

(iii) m is σ-additive.
(iv) m is completely additive.
(v) m is positive (negative).
(vi) m is a Jordan measure.
(vii) m is nondecreasing (nonincreasing).
(viii) m is nonnegative (nonpositive) for small positive values.
(ix) m is continuous in $[0, 1]$.
(x) m is bounded.
(xi) m is bounded on some interval $[a, b]$, where $0 \leq a < b \leq 1$.
(xii) m is continuous in some point $t_0 \in [0, 1]$.

Proof. The equivalence of (i), (v), (vi), (vii), and (viii) has been established above. Due to (2.1), (ix) and (i) are equivalent.

(ii) $\Rightarrow$ (v). The Q_1-regularity of m implies that given $t \in (0, 1)$, three is an increasing sequence of natural numbers $\{q_n\}$ with $q_n \leq t$ such that $m(t) = \lim_n m(q_n) = \lim_n q_n m(1) = t_0 m(t)$, where $t_0 = \lim_n q_n$. Then $m(t)$ and $m(1)$ have the same sign, so that m is either positive or negative.

The converse implication is evident.

(ii) and (iii) are equivalent, and similarly, (ii) and (iv).

(x) $\Rightarrow$ (ix). Let m be a bounded measure on $[0, 1]$. Take the three-dimensional Hilbert space R_3 and let $\mathcal{S}(R_3)$ be a unit sphere in R_3, that is, $\mathcal{S}(R_3) = \{x \in R_3: \|x\| = 1\}$. Fix a unit vector $e \in R_3$. Then the mapping $x \mapsto |(e, x)|^2$, $x \in \mathcal{S}(R_3)$, maps $\mathcal{S}(R_3)$ onto the interval $[0, 1]$. Indeed, if e_1 is a unit vector in R_3 which is orthogonal to e, then for unit vectors $x_\phi := \cos\phi\, e + \sin\phi\, e_1$, $\phi \in [0, \pi/2]$, we have $|(e, x_\phi)|^2 = \cos^2\phi$.

Define the mapping $f: \mathcal{S}(R_3) \to R$ via $f(x) := m(|(e, x)|^2)$, $x \in \mathcal{S}(R_3)$. Then f is a frame function in R_3, i.e., if x_1, x_2, x_3 and y_1, y_2, y_3 are two orthonormal bases in R_3, then $\Sigma_{i=1}^3 f(x_i) = \Sigma_{i=1}^3 f(y_i)$. The boundedness of m entails the boundedness of f. Using the Gleason theorem for finite-dimensional Hilbert spaces (Gleason, 1957; Dvurečenskij, 1993, Theorem 3.2.15), we find that f is continuous.

If now $t_n \to t$ in $[0, 1]$, then there exist ϕ_n, ϕ in $[0, \pi/2]$ such that $\phi_n \to \phi$, and $t_n \doteq |(e, x_{\phi_n})|^2 \to t = |(e, x_\phi)|^2$, so that $m(t_n) \to m(t)$ because $x_{\phi_n} \to x_\phi$. Therefore, m is continuous.

The converse statement is simple.

(xi) $\Rightarrow$ (x). First of all let $a = 0$, i.e., m is bounded on $[0, b]$, where $0 < b \leq 1$, so that there is a constant $K \geq 0$ such that $|m(s)| \leq K$ for any $s \in [0, b]$. There is an integer n_0 such that $1 \leq n_0 b$. Given $t \in (b, 1]$, there is an integer n, $1 \leq n \leq n_0$, such that $(n - 1)b < t \leq nb$. Then

$$m(t) = m[(n-1)b \oplus (t - (n-1)b)] = (n-1)m(b) + m(t - (n-1)b)$$
$$\leq n_0 K + K = (n_0 + 1)K$$

which means that m is bounded on $[0, 1]$.

Let now m be bounded on $[a, b]$, where $0 < a < b \leq 1$. There is an integer n_1 such that $a - n_1(b - a) \leq 0$. Given $t \in (0, a)$, there is an integer n, $1 \leq n \leq n_1$, such that $a - n(b - a) < t \leq a - (n - 1)(b - a)$. Hence $a < t + n(b - a) \leq b$ and $|m(t) + m(n(b - a))| \leq K$, so that $|m(t)| \leq K + n_1 |m(b - a)|$, which proves that m is bounded on $[0, b]$, and consequently m is bounded on $[0, 1]$.

(xii) → (xi). Suppose m to be continuous in some point $t_0 \in (0, 1)$. Given $\epsilon > 0$ there is a $\delta > 0$ such that, for any $t \in [0, 1]$ with $|t - t_0| < \delta$, we have $|m(t) - m(t_0)| < \epsilon$. So there is $\delta > 0$ such that m on $[t_0 - \delta, t_0 + \delta]$ is bounded. If now m is continuous in $t_0 = 0$ or in $t_0 = 1$, we find intervals $[0, \delta]$ or $[1 - \delta, 1]$ on which m is bounded. ■

Remark 2.2. Let ψ be a discontinuous additive functional on R (Hamel, 1905); then $m(t) := \psi(t)$, $t \in [0, 1]$, is an unbounded measure on $[0, 1]$.

3. CAUCHY'S FUNCTIONAL EQUATION

Cauchy's basic functional equation

$$f(x + y) = f(x) + f(y) \tag{3.1}$$

where $x, y \in$ R, was solved by Cauchy in 1821 (Aczél, 1966) and it was proved that if f is continuous everywhere on R, then

$$f(x) = cx \tag{3.2}$$

where c is a real constant. If we limit ourselves to equation (3.1), where we suppose that (3.1) holds in the triangle

$$0 \leq x \leq 1, \qquad 0 \leq y \leq 1, \qquad 0 \leq x + y \leq 1 \tag{3.3}$$

we see that any measure m on $[0, 1]$ induces a solution $f(x) := m(x)$, $x \in [0, 1]$. Conversely, any such solution of Cauchy's equation implies a measure m on $[0, 1]$.

In the fundamental work on functional equations (Aczél, 1966, Theorem 2.1.2) there is a partial solution of (3.1) with (3.3): If $f(x) \geq 0$ for small positive x values, then

$$f(x) = cx, \qquad x \in [0, 1] \tag{3.4}$$

Using measures on $[0, 1]$, we give the following solution of functional equation (3.1) with (3.3).

Theorem 3.1. Let f be a solution of Cauchy's functional equation (3.1) with (3.3). The following statements are equivalent:

(i) f has the form (3.4).

(ii) f is continuous from the left on rational numbers in each point of [0, 1].
(iii) f is nonnegative (nonpositive).
(iv) $f = f_1 - f_2$, where f_1, f_2 are nonnegative solutions of (3.1) with (3.3).
(v) f is nondecreasing (nonincreasing).
(vi) f is nonnegative (nonpositive) for small positive values.
(vii) f is continuous in [0, 1].
(viii) f is bounded.
(ix) f is bounded on some interval $[a, b]$, where $0 \leq a < b \leq 1$.
(x) f is continuous in some point $t_0 \in [0, 1]$.

Proof. It follows from Theorem 2.1. ∎

Let now Cauchy's functional equation $f(x + y) = f(x) + f(y)$, $x, y \in [0, 1]$ with $x + y \in [0, 1]$, be assumed to be X-valued, where X is a Banach space.

Theorem 3.2. Let X be a Banach space and f be an X-valued solution of Cauchy's functional equation (3.1) with (3.3). The following statements are equivalent:
(i) f is bounded, i.e., $\sup\{\|f(x)\|: x \in [0, 1]\} < \infty$.
(ii) f is continuous.
(iii) $f(x) = cf(1)$, $x \in [0, 1]$.

Proof. (i) $\Rightarrow$ (iii). Let f be an X-valued solution of (3.1) with (3.3). Take an arbitrary continuous linear functional ψ on X. Then $\psi \circ f$ is a bounded complex-valued solution of Cauchy's basic functional equation on [0, 1]. Assuming separately the real and imaginary parts of $\psi \circ f$, we see that $\psi \circ f$ is continuous. Hence, by Theorem 3.1,

$$\psi(f(x)) = (\psi \circ f)(x) = x(\psi \circ f)(1) = \psi(xf(1))$$

for any $x \in [0, 1]$. Because all bounded functionals on X separate the points of X, $f(x) = xf(1)$ for any $x \in [0, 1]$.

All other implications are simple. ∎

Assume now that on the interval $L = [0, 1]$ there is a general partial binary operation $\oplus$ satisfying (EAi)–(EAiv). If g: $[0, 1] \to [0, 1]$ with $g(0) = 0$, $g(1) = 1$, is a continuous, strictly increasing function, we define $a \oplus_g b$ for $a, b \in [0, 1]$ iff $g(a) + g(b) \leq 1$, and then

$$a \oplus_g b := g^{-1}(g(a) + g(b)) \tag{3.5}$$

An easy calculation shows that [0, 1] with 0, 1, and $\oplus_g$ is an effect algebra. Conversely (Mesiar, 1994), any effect algebra [0, 1] with 0, 1, and $\oplus$ is of the form (3.5), and g is uniquely determined.

Theorem 3.3. Let f be a solution to Cauchy's functional equation

$$f(x \oplus_g y) = f(x) + f(y) \tag{3.6}$$

for $x, y, x \oplus_g y \in [0, 1]$, where $\oplus_g$ is induced by (3.5). Then the statement

$$f(x) = g(x)f(1), \qquad x \in [0, 1] \tag{3.7}$$

is equivalent to any statement (ii)–(x) in Theorem 3.1.

Proof. Suppose that f is any solution to (3.6). Define a new real-valued function $\hat{f}(x)$: $[0, 1] \to R$ via $\hat{f}(x) = f((g^{-1}(x))$, $x \in [0, 1]$. Then $\hat{f}(x + y) = \hat{f}(x) + \hat{f}(y)$ for all $x, y \in [0, 1]$ with $x + y \in [0, 1]$. If, for example, f is assumed to be bounded, so is $\hat{f}$, and by Theorem 3.1, $\hat{f}(x) = x\hat{f}(1)$. Hence $\hat{f}(x) = f(g^{-1}(x)) = x\hat{f}(1)$ and

$$\hat{f}(g(x)) = f((g^{-1}(g(x))) = f(x) = g(x)\hat{f}(1) = g(x)f(1) \quad \blacksquare$$

Theorem 3.4. Let X be a Banach space and let f be a solution of Cauchy's X-valued functional equation

$$f(x \oplus_g y) = f(x) + f(y)$$

for $x, y \in [0, 1]$ with $x \oplus_g y \in [0, 1]$, where $\oplus_g$ is defined by (3.5). The following statements are equivalent:

(i) $f(x) = g(x)f(1)$, $x \in [0, 1]$.

(ii) f is continuous.

(iii) f is bounded, i.e., $\sup\{\|f(x)\|: x \in [0, 1]\} < \infty$.

Proof. It follows the same idea as that of Theorem 3.3. ■

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